The sum of two jointly stationary processes is stationary.

proof: Suppose $\{y_t\}$ & $\{z_t\}$ are stationary.

Let $x_t = y_t + z_t$

$$E[x_t] = E[y_t] + E[z_t] = \mu_y + \mu_z$$

$$\begin{aligned}
cov(x_s, x_t) &= cov(y_s + z_s, y_t + z_t) \\
&= cov(y_s, y_t) + cov(y_s, z_t) + cov(z_s, y_t) \\
&\quad + cov(z_s, z_t)
\end{aligned}$$

Let $h = t - s$

$$\begin{aligned}
&= \gamma_y(h) + \gamma_{yz}(h) + \gamma_{zy}(h) + \gamma_z(h) \\
&= \gamma_y(h) + \gamma_{yz}(h) + \gamma_{yz}(-h) + \gamma_z(h)
\end{aligned}$$

Recall: $\gamma_y(h) = \gamma_y(-h)$

$\gamma_{xy}(h) = \gamma_{yx}(-h)$

$E[x_t]$ is constant & $cov(x_s, x_t)$ depends on h, but not t,

So, x_t is stationary.

Suppose $\{X_t\}$ has a linear time trend,

$$X_t = \alpha + \beta t + y_t ,$$

where y_t is stationary.

Then $\{\nabla X_t\}$ is stationary.

proof:

$$\begin{aligned}\nabla X_t &= X_t - X_{t-1} \\ &= (\alpha + \beta t + y_t) - (\alpha + \beta \cdot (t-1) + y_{t-1}) \\ &= \beta + y_t - y_{t-1} \\ &= \beta + \nabla y_t\end{aligned}$$

Since y_t is stationary then so is ∇y_t.

Thus so is ∇X_t.

<u>Key point</u>:

If $X_t = \alpha + \beta t + y_t$, where y_t is stationary.

Then $X_t - (\alpha + \beta t)$ is stationary

and so is ∇X_t

Now consider a random walk with drift.

$$x_t = \delta + x_{t-1} + w_t$$

This is not stationary. We could try to estimate a linear time trend from data, but this would not give stationarity.

But $\nabla x_t = x_t - x_{t-1} = \delta + w_t$

$\uparrow$ stationary.

Backshift operator

$$B x_t = x_{t-1}$$

$$B^2(x_t) = B(B(x_t)) = B x_{t-1} = x_{t-2}$$

$$B^k(x_t) = \cdots = x_{t-k}$$

$$B^{-1}(x_t) = x_{t+1}$$

etc.

$\vdots$

$$(1-B)x_t = x_t - Bx_t = x_t - x_{t-1}$$

$\uparrow$ this is the difference operator.

$$(1-B)^2 x_t = (1-B)\cdot \nabla x_t = \nabla^2 x_t$$

$$= (1-2B + B^2)\, x_t$$

$$= x_t - 2x_{t-1} + x_{t-2}$$

$$= (x_t - x_{t-1}) - (x_{t-1} - x_{t-2})$$

$$= \nabla x_t - \nabla x_{t-1}$$

$$= \nabla^2 x_t$$

transform data

Frequently it is useful to do some transformation of the data to get stationarity.

Log( ) is a common choice.

The Box-Cox family is of form:

$$y_t = \begin{cases} (x_t^{\lambda} - 1)/\lambda & \lambda \neq 0 \\ \log x_t & \lambda = 0 \end{cases}$$