Exercise 3.1 (Solution)

(1) $y_t = 0.3\, y_{t-1} + w_t - 0.8\, w_{t-1} + 0.15\, w_{t-2}$

$$\varphi(B)\, y_t = \theta(B)\, w_t$$

$$\varphi(z) = \cancel{1 - 0.3z}$$

no roots

$$\theta(z) = 1 - 0.8z + 0.15z^2$$

$$= (1 - 0.5z)\cancel{(1 - 0.3z)}$$

Root at $z = -2$.

Parameter redundancy. The model is equivalent to

$$y_t = w_t - 0.5\, w_{t-1}$$

It is causal & invertible

$$\psi(z) = 1 - 0.5z$$

$$\pi(z) = \sum_{i=0}^{\infty} (-\tfrac{1}{2})^i z^i$$

Exercise 3.1 (Solution)

(2) $y_t = -\frac{5}{6} y_{t-1} - \frac{1}{6} y_{t-2} + w_t - \frac{1}{2} w_{t-1}$

$$\varphi(B)\, y_t = \theta(B)\, w_t$$

$$\phi(z) = 1 + \tfrac{5}{6} z + \tfrac{1}{6} z^2$$
$$= \left(1 + z/2\right)\left(1 + z/3\right)$$

Roots are $z = -2, -3$. Causal.

$$\theta(z) = 1 - \tfrac{1}{2} z$$

Root is 2. Invertible.

$MA(\infty)$. We seek $\psi(B)$ such that

$$\varphi(B)\,\psi(B)\,w_t = \theta(B)\,w_t$$

$$\left(1+\tfrac{5}{6}z+\tfrac{1}{6}z^2\right)\left(1+\psi_1 z+\psi_2 z^2+\psi_3 z^3+\cdots\right) = 1-\tfrac{1}{2}z$$

z: $\psi_1 + 5/6 = -1/2 \qquad \psi_1 = -4/3$

z^2: $\psi_2 + 5/6\,\psi_1 + 1/6 = 0 \qquad \psi_2 = 17/18$

z^3 $\psi_3 + 5/6\,\psi_2 + 1/6\,\psi_1 = 0 \qquad \psi_3 = -61/108$

$\vdots$

z^n: $\psi_n + 5/6\,\psi_{n-1} + 1/6\,\psi_{n-2} = 0$

AR(∞). We see $\pi(B)$ such that

$$\phi(B)\, y_t = \theta(B)\, \pi(B)\, y_t$$

$$1 + \tfrac{5}{6} z + \tfrac{1}{6} z^2 = \left(1 - \tfrac{1}{2} z\right)\left(1 + \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + \cdots\right)$$

z: $\pi_1 - \tfrac{1}{2} = \tfrac{5}{6}$ $\qquad \pi_1 = 4/3$

z^2: $\pi_2 - \tfrac{1}{2}\pi_1 = \tfrac{1}{6}$ $\qquad \pi_2 = 5/6$

z^3: $\pi_3 - \tfrac{1}{2}\pi_2 = 0$ $\qquad \pi_3 = 5/12$

$\vdots$

z^n $\quad \pi_n - \frac{1}{2}\pi_{n-1} = 0$

Check

ARMAtoMA(ar = c(-5/6, -1/6), ma = -1/2,
lag.max = 5)

ARMAtoMA(ar = 1/2, ma = c(5/6, 1/6),
lag.max = 5)