Autoregression

$$x_t = \alpha_0 + \alpha_1 x_{t-1} + \cdots + \alpha_p x_{t-p} + w_t$$

where $w_t \sim WN(\sigma_w^2)$

Example

$$x_t = x_{t-1} - 0.9\, x_{t-2} + w_t$$

where $w_t \sim WN(0.01)$

Random Walk

$$x_t = x_{t-1} + w_t$$

where $w_t \sim WN(\sigma_w^2)$

$x_0 = X_0$

$$x_t = \underbrace{(x_{t-2} + w_{t-1})}_{x_{t-1}} + w_t$$

$$x_t = ((x_{t-3} + w_{t-2}) + w_{t-1}) + w_t$$

$$= (((x_{t-4} + w_{t-3}) + w_{t-2}) + w_{t-1}) + w_t$$

$$\vdots$$

$$= X_0 + \sum_{j=1}^{t} w_t$$

Random walk with drift

$$X_t = \delta + x_{t-1} + w_t$$

where $w_t \sim WN(\sigma_w^2)$

$x_0 = X_0$

$$x_t = \delta + (\delta + x_{t-2} + w_{t-1}) + w_t.$$

$$= \delta + (\delta + (\delta + x_{t-3} + w_{t-2}) + w_{t-1}) + w_t$$

$$\vdots$$

$\vdots$

$$x_t = \delta t + \sum_{i=1}^{t} w_i$$

drift

Signal plus noise

$$y_t = s_t + w_t$$

signal

noise

Example

$$s_t = A\cos(2\pi\omega t + \varphi)$$

A = amplitude

ω = frequency

φ = shift

Try this with $A = 2$, $\omega = 1/50$ $\varphi = 0.6\pi$

Given a time series $\{x_1, x_2 \dots x_t\}$

Recall that a complete description involves the joint CDF

$$F(x_1, x_2 \cdots x_t).$$

But frequently in practice it is sufficient to just be concerned with means & covariances.

Def: The mean function is

$$\mu_{xt} = E[x_t]$$

Example

$$x_t = \sin(2\pi t/50) + w_t$$

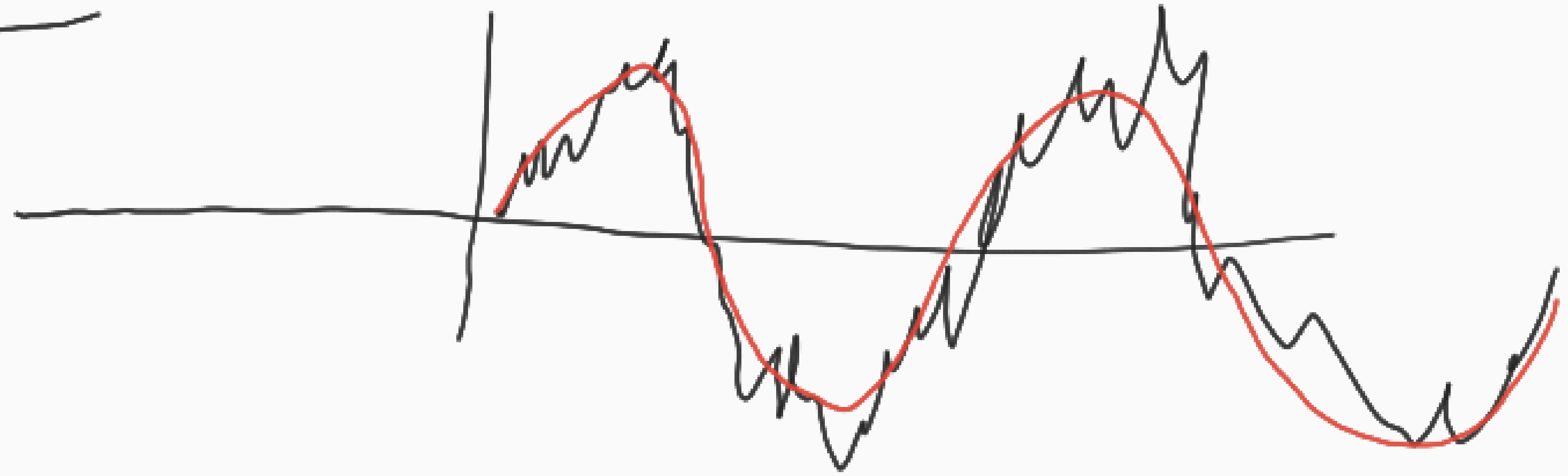
$$\mu_{xt} = \sin(2\pi t/50)$$

If there is no ambiguity, omit the subscript x.

$$\mu_t = \sin(2\pi t/50)$$

$s_t = \mu_t$ ———

x_t ———



Def: The autocovariance function is

$$\gamma_x(s,t) = \operatorname{cov}(X_s, X_t) = E\left[(X_s - \mu_s)(X_t - \mu_t)\right]$$

Autocovariance measures linear dependence between two observations of same series taken at different points in time.

Remember: Two random variables can be dependent even if they are uncorrelated!

Note that $\gamma_x(t,t) = \operatorname{cov}(X_t, X_t) = Var(X_t)$

Def: The autocorrelation fn is

$$\rho_x(s,t) = \frac{\gamma_x(s,t)}{\sqrt{\gamma_x(s,s)\,\gamma_x(t,t)}}$$

Example

$$x_t = 2\sin(2\pi t/50) + w_t$$

$$\mu_t = 2\sin(2\pi t/50)$$

$$\gamma_x(s,t) = E\left[(x_s - \mu_s)(x_t - \mu_t)\right]$$

$$= E\left[w_s \cdot w_t\right] = \begin{cases} \sigma_w^2 & s = t \\ 0 & \text{otherwise} \end{cases}$$

Def: $\{X_1, X_2, \ldots\}$ is (weakly) stationary if

$\mu_t = \mu_s$ for $s, t = 1, 2, \ldots$

and

$\gamma(s,t) = \gamma(s+h, t+h)$

for all $s, t = 1, 2, \ldots$
and $h > 0$.

In this case we can simply write

$\mu = E[X_t]$

$\gamma_h = \gamma(t, t+h)$

Don't need to indicate time t