Moving Average

$$X_t = \frac{1}{3}\left(w_{t-1} + w_t + w_{t+1}\right), \quad w_t \sim WN(\sigma_w^2)$$

$$\mu_t = E\left[X_t\right] = \frac{1}{3}E\left[w_{t-1} + w_t + w_{t+1}\right] = 0$$

$$\gamma(s,t) = \mathrm{cov}\left[(X_s - \mu_s)(X_t - \mu_t)\right] =$$

$$= \frac{1}{9}E\left[(w_{s-1} + w_s + w_{s+1})(w_{t-1} + w_t + w_{t+1})\right]$$

$$= \begin{cases} \frac{3}{9}\sigma_w^2 & |s-t| = 0 \\ \frac{2}{9}\sigma_w^2 & |s-t| = 1 \\ \frac{1}{9}\sigma_w^2 & |s-t| = 2 \\ 0 & \text{else} \end{cases}$$

Alternatively,

write the model as a linear process:

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$$

where $\psi_{-1} = 1/3$

$\psi_0 = 1/3$

$\psi_1 = 1/3$

$\psi_j = 0$ else

$E[X_t] = \mu$

$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h} \psi_j = \begin{cases} 3/9\, \sigma_w^2 & h = 0 \\ 2/9\, \sigma_w^2 & |h| = 1 \\ 1/9\, \sigma_w^2 & |h| = 2 \\ 0 & \text{else} \end{cases}$$

Autoregressive Model

Example: $y_t = \alpha + \varphi y_{t-1} + w_t$, $w_t \sim WN(\sigma_w^2)$

and assume $|\varphi| < 1$.

ϕ

For the moment, assume stationarity.

$$E[y_t] = \alpha + \varphi E[y_{t-1}] + 0$$

$$\mu = \alpha + \varphi \cdot \mu \qquad \text{(because of stationarity)}$$

$$\mu - \varphi\mu = \alpha$$

$$\mu(1-\varphi) = \alpha$$

$$\mu = \alpha / 1-\varphi$$

$$\gamma_0 = Var(y_t) = \varphi^2 Var(y_{t-1}) + \sigma_w^2$$

$$\gamma_0 = \varphi^2 \gamma_0 + \sigma_w^2 \qquad (\text{stationarity})$$

$$\gamma_0 (1 - \varphi^2) = \sigma_w^2$$

$$\gamma_0 = \sigma_w^2 / (1 - \varphi^2)$$

$$\gamma_1 = cov(y_{t-1}, \underline{y_t}) = cov(y_{t-1}, \underline{\varphi y_{t-1} + w_t})$$

$$\gamma_1 = \varphi \gamma_0$$

γ_2: $$cov(y_{t-2}, \underline{y_t}) = cov(y_{t-2}, \underline{\varphi y_{t-1} + w_t})$$

$$\gamma_2 = \varphi \gamma_1 = \varphi^2 \gamma_0$$

etc ...

Now let's do the same thing using the linear process formulation.

$$y_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$$

where $\psi_j = \varphi^j \quad j = 0, 1, 2, \ldots$

and $\psi_j = 0$ else.

Notice that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

if and only if $|\varphi| < 1$.

$$\gamma(h) = \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

$$= \sigma_w^2 \sum_{j=0}^{\infty} \left(\varphi^j \cdot \varphi^{j+h} \right)$$

$$= \sigma_w^2 \sum_{j=0}^{\infty} \varphi^{2j+h} = \frac{\sigma_w^2}{1-\varphi^2} \cdot \varphi^h$$

$$= \varphi^h \cdot \gamma(0) .$$

Estimation of mean

Need stationary model in order to be able to do anything:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Estimation of covariances

$$\hat{\gamma}(-h) = \hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (X_{t+h} - \hat{\mu})(X_t - \hat{\mu})$$

Estimation of autocorrelations

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \qquad h = \cdots -2, -1, 0, 1, 2, \cdots$$

Theorem: Under general conditions, for a white noise process, $\{w_t\}$,

$$\sqrt{n}\left(\hat{\rho}(h) - 0\right) \xrightarrow{D} N(0,1)$$

for all h.

So, we can test the hypothesis

$$H_0: \rho(h) = 0$$

by checking whether $\left|\hat{\rho}(h)\right| < \pm \frac{2}{\sqrt{n}}$

Or, a joint test (for many values of h),

$$n \cdot \sum_{h=1}^{H} |\hat{\rho}(h)^2| \xrightarrow{D} \chi^2_H$$

In R: given x[1] ... x[T]

```
xbar = mean(x)
cov = acf(x)
```