Problem 1.13

$$x_t = w_t \quad , \quad w_t \sim wn(\sigma_w^2)$$

$$y_t = w_t - \theta w_{t-1} + u_t \qquad u \sim wn(\sigma_u^2)$$

(a) $\gamma_y(0) = E\left[(w_t - \theta w_{t-1} + u_t)^2\right] = (1+\theta^2)\sigma_w^2 + \sigma_u^2$

$\gamma_y(\pm 1) = E\left[(w_t - \theta w_{t-1} + u_t)(w_{t-1} - \theta w_{t-2} + u_{t-1})\right]$

$= -\theta \sigma_w^2$

$\gamma_y(h) = 0$ ~~otherwise~~ $|h| > 1$

$\rho_y(h) = \gamma_y(h) / \gamma_y(0)$

(b) $\gamma_{xy}(0) = E\left[w_t (w_t - \theta w_{t-1} + u_t)\right] = \sigma_w^2$

$\gamma_{xy}(-1) = E\left[w_{t-1}(w_t - \theta w_{t-1} + u_t)\right] = -\theta \sigma_w^2$

$\gamma_{xy}(h) = 0$ otherwise

(c) $E[x_t] = 0$, $E[y_t] = 0$ for all t.

The autocovariances & crosscovariances all depend on h but not t.

1.15

$$x_t = w_t w_{t-1}$$

$$E[x_t] = E[w_t w_{t-1}] = 0$$

$$\gamma_0 = E[x_t^2] = E[w_t^2 w_{t-1}^2] = E[w_t^2]\, E[w_{t-1}^2]$$
$$= \sigma_w^4$$

$$\gamma_1 = E[x_{t+1} \cdot x_t] = E[w_{t+1} \cdot w_t \cdot w_t \cdot w_{t-1}]$$
$$= E[w_t^2] \cdot E[w_{t+1}] \cdot E[w_{t-1}] = 0$$

etc

$$\gamma_h = \begin{cases} \sigma_w^4 & h = 0 \\ 0 & \text{else} \end{cases}$$

it is stationary.

Problem 2.6

(a) $\mu_x(t) = \beta_0 + \beta_1 t$.

If $\beta_1 \neq 0$, this depends on t, so $\{x_t\}$ is not stationary.

(b)
$$\begin{aligned}\nabla x_t &= x_t - x_{t-1} \\ &= (\beta_0 + \beta_1 t + w_t) - (\beta_0 + \beta_1 (t-1) + w_{t-1}) \\ &= \beta_1 + w_t - w_{t-1}\end{aligned}$$

$$\mu_{\nabla x}(t) = \beta_1$$

$$\gamma_{\nabla x}(t, t+h) = \begin{cases} 2\sigma_w^2 & h = 0 \\ -\sigma_w^2 & h = \pm 1 \\ 0 & |h| > 1 \end{cases}$$

The mean & covariance do not depend on t.

2.6 continued

(c) $\nabla X_t = \beta_1 + y_t - y_{t-1}$

$$E\left[\nabla X_t\right] = \beta_1 + \mu_y - \mu_y = \beta_1$$

$$\mathrm{Cov}\left[\nabla X_t, \nabla X_{t+h}\right] = E\left[(y_t - y_{t-1})(y_{t+h} - y_{t+h-1})\right]$$

$$= 2\gamma_y(h) - \gamma_y(h-1) - \gamma_y(h+1)$$

Mean & Covariance do not depend on t.