problem 3.2

(a) $x_0 = w_0$

$$x_1 = \varphi x_0 + w_1 = \varphi w_0 + w_1 = \sum_{j=0}^{1} \varphi^j w_{1-j}$$

$$X_2 = \varphi X_1 + w_2 = \varphi \sum_{j=0}^{1} \varphi_j w_{1-j} + w_2$$

$$= \sum_{j=0}^{2} \varphi_j w_{2-j}$$

$\vdots$

(b) $$E[X_t] = E\left[\sum_{j=0}^{t} \varphi^j w_{t-j}\right] = \sum_{j=0}^{t} \varphi^j E[w_{t-j}]$$

$$= 0$$

Linearity of expectation

(c) $$Var[X_t] = Var\left[\sum_{j=0}^{t} \varphi^j w_{t-j}\right] = \sum_{j=0}^{t} \varphi^{2j} \, Var \, w_{t-j}$$

$$= \sigma_w^2 \cdot \sum_{j=0}^{t} \varphi^{2j} = \sigma_w^2 \frac{1-\varphi^{2(t+1)}}{1-\varphi^2}$$

geometric series

3.2 continued

(d) $$\operatorname{cov}(x_{t+h}, t) = E\left[\left(\sum_{j=0}^{t+h} \varphi^j w_{t+h-j}\right)\left(\sum_{i=0}^{t} \varphi^i w_{t-i}\right)\right]$$

$$= E\left[\left(\sum_{j=-h}^{t} \varphi^{j+h} w_{t-j}\right)\left(\sum_{i=0}^{t} \varphi^i w_{t-i}\right)\right]$$

$$= E\left[\sum_{i=0}^{t} \varphi^{2i+h} w_{t-i}\right]$$

$$= \varphi^h E\left[\sum_{i=0}^{t} \varphi^{2i} w_{t-i}\right] = \varphi^h \operatorname{Var} X_t .$$

(e) X_t is not stationary because $\operatorname{Var}(X_t)$ & $\operatorname{cov}(X_t, X_{t+h})$ depend on t.

(f) $$\lim_{t\to\infty} \operatorname{Var}(x_t) = \lim_{t\to\infty} \sigma_w^2 \frac{1-\varphi^{2(t+1)}}{1-\varphi^2}$$

$$= \frac{\sigma_w^2}{1-\varphi^2} .$$

3.2 continued

(g) We could simulate $\{X_t\}_{-k}^{n}$ for some large integer k & throw away $X_{-k}, \cdots, X_{-1}$

(h) $\mathrm{Var}\, X_0 = \sigma_w^2 / 1-\varphi^2$

$$\mathrm{Var}\, X_1 = \varphi^2 \mathrm{Var}\, X_0 + \sigma_w^2$$

$$= \frac{\sigma_w^2 \varphi^2}{1-\varphi^2} + \sigma_w^2$$

$$= \sigma_w^2 \left[1 + \frac{\varphi^2}{1-\varphi^2} \right]$$

$$= \sigma_w^2 \left[\frac{1}{1-\varphi^2} \right] = \mathrm{Var}\, X_0 .$$

Similarly for $X_2, X_3 \ldots$