Ch 3 ARIMA Models

- Recall last week: Box-Jenkins Methodology.
  (see also Sec. 3.1 of book).

Example AR(1)

$$y_t = \varphi y_{t-1} + w_t, \quad w_t \sim WN(\sigma_w^2)$$

$$t = 0, \pm 1, \pm 2, \ldots \qquad |\varphi| < 1$$

Recall:

$$\begin{aligned}
y_t &= \varphi(\varphi y_{t-2} + w_{t-1}) + w_t \\
&= \varphi^2 y_{t-2} + \varphi w_{t-1} + w_t \\
&= \varphi^2(\varphi y_{t-3} + w_{t-2}) + \varphi w_{t-1} + w_t \\
&= \varphi^3 y_{t-3} + \varphi^2 w_{t-2} + \varphi w_{t-1} + w_t \\
&= \sum_{i=0}^{\infty} \varphi^i w_{t-i}
\end{aligned}$$

Forecasting

$$\begin{aligned}
\hat{y}_{t+1|t} &= \varphi y_t + E[w_{t+1}] \\
&= \varphi y_t
\end{aligned}$$

$$\begin{aligned}
\hat{y}_{t+2|t} &= \varphi^2 y_t + E[w_{t+2}] + \varphi E[w_{t+1}] \\
&= \varphi^2 y_t
\end{aligned}$$

$\vdots$

$$\hat{y}_{t+k|t} = \varphi^k y_t$$

Assume $h > 0$

$$\gamma(-h) = \gamma(h) = \operatorname{cov}(y_{t+h}, y_t)$$

$$= E\left[\left(\sum_{i=0}^{\infty} \varphi^i w_{t+h-i}\right)\left(\sum_{j=0}^{\infty} \varphi^j w_{t-j}\right)\right]$$

Let $k = i-h$

$$= E\left[\left(\sum_{k=-h}^{\infty} \varphi^{k+h} w_{t-k}\right)\left(\sum_{j=0}^{\infty} \varphi^j w_{t-j}\right)\right]$$

$$= E\left[\sum_{k=-h}^{\infty}\sum_{j=0}^{\infty} \varphi^{k+h} \varphi^j w_{t-k} \cdot w_{t-j}\right]$$

these are all zero unless $k = j$

$$= E\left[\sum_{j=0}^{\infty} \varphi^{j+h} \varphi^j w_{t-j}^2\right]$$

$$= \sum_{j=0}^{\infty} \varphi^{2j+h} \cdot \sigma_w^2$$

$$= \boxed{\sigma_w^2 \frac{\varphi^h}{1-\varphi^2}}$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \varphi^h$$

Also, $\mu = E[y_t] = 0$.

Example

$$Z_t = \underset{\substack{\uparrow \\ \text{constant}}}{\alpha} + \varphi Z_{t-1} + w_t$$

$$E[Z_t] = \alpha + \varphi E[Z_{t-1}] + E[w_t]$$

$$\mu = \alpha + \varphi \cdot \mu$$

$$\mu = \frac{\alpha}{1-\varphi}$$

$\{y_t\}$ & $\{Z_t\}$ have the same ACF.

Note that any AR(1) model can be transformed to one that is mean zero by

$$y_t = Z_t - \mu$$

We will frequently assume $\alpha = 0$ without loss of generality.

Def: The AR(p) is of form

$$y_t = \alpha + \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + w_t$$

$$E[y_t] = \alpha + \varphi_1 E[y_{t-1}] + \cdots + \varphi_p E[y_{t-p}] + \cancel{E[w_t]}$$

Assuming stationarity,

$$\mu = \alpha + \varphi_1 \mu + \cdots + \varphi_p \mu$$

$$\mu = \frac{\alpha}{1 - \varphi_1 - \varphi_2 - \cdots \varphi_p}$$

The demeaned model is

$$Z_t = y_t - \mu$$
$$= \varphi_1 Z_{t-1} + \cdots + \varphi_p Z_{t-p} + w_t .$$

Recall, we can also write this as

$$Z_t - \varphi_1 Z_{t-1} - \varphi_2 Z_{t-2} - \cdots - \varphi_p Z_{t-p} = w_t$$

$$(1 - \varphi_1 B - \varphi_2 B^2 - \cdots - \varphi_p B^p) Z_t = w_t$$

$$\varphi(z) = 1 - \varphi_1 z - \cdots - \varphi_p z^p$$

is referred to as a "lag polynomial"

&

$$\varphi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p$$

is referred to as the autoregressive operator

Explosive AR models

Example: AR(1) with $|\varphi| > 1$ is explosive.

Recall that a linear process is one that can be written in form:

$$y_t = \mu + \sum_{i=-\infty}^{\infty} \psi_i w_{t-i}$$

where $\sum_{i=-\infty}^{\infty} |\psi_i| < \infty$.

Recall also that we can write the AR(1) model as $y_t = \sum_{i=0}^{\infty} \varphi^i y_{t-i}$

So, $\{y_t\}$ is a linear process if & only if $|\varphi| < 1$.

Now suppose $|\varphi| > 1$.

$$y_t = \varphi y_{t-1} + w_t$$

$$y_{t-1} = \frac{1}{\varphi} y_t - \frac{1}{\varphi} w_t$$

$$= \frac{1}{\varphi}\left[\frac{1}{\varphi} y_{t+1} - \frac{1}{\varphi} w_{t+1}\right] - \frac{1}{\varphi} w_t$$

$$= \frac{1}{\varphi^2} y_{t+1} - \frac{1}{\varphi^2} w_{t+1} - \frac{1}{\varphi} w_t$$

$$= \quad \vdots$$

$$= -\sum_{j=1}^{\infty} \varphi^{-j} w_{t+j}$$

Since $\left|\sum_{j=1}^{\infty} \varphi^{-j}\right| < \infty$, this shows that this model can be written a linear process.

recall, $y_t = \varphi y_{t-1} + w_t$ is <u>causal</u> if & only if $|\varphi| < 1$.

We are almost always interested in causal models.

<u>Method of matching coefficients</u>

Given the AR(p) model

(1) $$\varphi(B) y_t = w_t$$

we seek to write this in linear process form

(2) $$y_t = \psi(B) w_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

For the AR(1) model this was easy.

In general, do as follows.

Substitute (2) into (1)

$$\varphi(B)\,\psi(B)\,w_t = w_t$$

So, $\varphi(z)\,\psi(z) = 1$

$$(1-\varphi_1 z - \varphi_2 z^2 \cdots - \varphi_p z^p)(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \psi_4 z^4 + \cdots) = 1$$

1: $1 \cdot 1 = 1$ $\quad\quad \psi_0 = 1$

z: $\psi_1 z - \varphi_1 z = 0$ $\quad$ so, $\psi_1 = \varphi_1$

z^2: $\psi_2 - \varphi_1 \psi_1 - \varphi_2 = 0$ $\quad$ so $\psi_2 = \varphi_1 \psi_1 + \varphi_2$

z^3: $\psi_3 - \varphi_1 \psi_2 - \varphi_2 \psi_1 - \varphi_3 = 0$ $\quad$ so, $\psi_3 = \varphi_1 \psi_2 + \varphi_2 \psi_1 + \varphi_3$

$\vdots$

$$y_t = \mu + \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

is sometimes referred to as the "Wold" form

or $MA(\infty)$.

__Exercise__: Write

$$y_t = 0.5 y_{t-1} + 0.3 y_{t-2} + w_t$$

in Wold form.

given $\varphi(B)\, y_t = w_t$

where $\varphi(z) = 1 - 0.5z - 0.3z^2$

We seek $y_t = \psi(B)\, w_t$

Substituting, we get

$$\varphi(B)\,\psi(B)\, w_t = w_t$$

So,

$$\varphi(z)\,\psi(z) = 1$$

$$(1 - 0.5z - 0.3z^2)(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots) = 1$$

z: $-0.5 + \psi_1 = 0$ so, $\psi_1 = 0.5$

z^2: $-0.3 - 0.5\psi_1 + \psi_2 = 0$ so, $\psi_2 = 0.5\psi_1 + 0.3$

$= 0.5 \cdot 0.5 + 0.3$

$= 0.55$

z^3: $-0.3\psi_1 - 0.5\psi_2 + \psi_3 = 0$ so, $\psi_3 = 0.5\psi_2 + 0.3\psi_1$

$= 0.425$

$\vdots$

z^k: $-0.3\psi_{k-2} - 0.5\psi_{k-1} + \psi_k = 0$

$\psi_k = 0.5\psi_{k-1} + 0.3\psi_{k-2}$