Moving average models

Example : MA(1)

$$y_t = w_t + \theta w_{t-1}$$

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2 & h=0 \\ \theta \sigma_w^2 & h=\pm 1 \\ 0 & \text{else} \end{cases}$$

$$\rho(h) = \begin{cases} 1 & h=0 \\ \theta / 1+\theta^2 & h=\pm 1 \\ 0 & \text{else} \end{cases}$$

$$\mu_t = E[y_t] = E[w_t] + \theta E[w_{t-1}]$$
$$= 0$$

It is stationary

MA(q)

$$y_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

The MA(q) model is a linear process:

$$y = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$$

where $\psi_j = \begin{cases} 1 & \text{if } j=0 \\ \theta_j & j=1,\dots,q \\ 0 & \text{else} \end{cases}$

note that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

for ~~any~~ any parameters $\theta_1, \cdots, \theta_q$

Also, note that it is always causal.

Non-uniqueness of MA model & non-invertibility

$$x_t = w_t + 0.2\, w_{t-1} \quad , \text{where } w_t \sim WN(1)$$

$$y_t = v_t + 5\, v_{t-1} \quad \text{where } v_t \sim WN(1/5)$$

They have the same mean & ACF. If $\{w_t\}$ & $\{v_t\}$ are both Gaussian WN, $\{x_t\}$ & $\{y_t\}$ are identical.

For identifiability, we would like to specify one of these as being of interest.

Analogous to what we did with the AR model, we will choose the one with the $AR(\infty)$ representation, i.e.,

$$w_t = \sum_{i=0}^{\infty} \pi_i\, y_{t-i}$$

where $\pi_0 = 1$ & $\sum_{i=0}^{\infty} |\pi_i| < \infty$.

Example MA(1)

given $y_t = w_t + \theta w_{t-1}$

write this as $y_t = \theta(B) w_t$

where $\theta(B) = 1 + \theta_1 B$

We seek $\pi_0, \pi_1, \pi_2, \ldots$ such that

$$\pi(B) x_t = w_t .$$

Solving recursively (as we did with AR(1))

we get

$$\pi(B) = \sum_{j=0}^{\infty} (-\theta)^j B^j$$

$$x_t + -\theta x_{t-1} + \theta^2 x_{t-2} - \theta^3 x_{t-3} \cdots = w_t$$

This converges iff $|\theta| < 1$.

We ~~as~~ refer to an MA(q) model with this property as being "invertible."

In general, we need to do matching coefficients to find the π_i.

Example MA(2)

given $X_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}$

(1) $\qquad = \theta(B)\, w_t$

where $\theta(B) = 1 + \theta_1 B + \theta_2 B^2$

We seek $\pi_0, \pi_1, \cdots$ such that

(2) $\qquad \pi(B)\, X_t = w_t$

where: $\pi(B) = 1 + \pi_1 B + \pi_2 B^2 + \cdots$

First, substitute (2) into (1)

$$X_t = \theta(B)\,\pi(B)\, X_t$$

so,

$$\theta(z)\,\pi(z) = 1 \qquad \text{for all } z.$$

$$(1 + \theta_1 z + \theta_2 z^2)(1 + \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + \cdots) = 1$$

z: $\theta_1 + \pi_1 = 0$ so $\pi_1 = -\theta_1$

z^2: $\theta_2 + \theta_1 \pi_1 + \pi_2 = 0$ so $\pi_2 = -\theta_1 \pi_1 - \theta_2$

z^3: $\theta_2 \pi_1 + \theta_1 \pi_2 + \pi_3 = 0$ so $\pi_3 = -\theta_1 \pi_2 - \theta_2 \pi_1$

$\vdots$

z^k: $\theta_2 \pi_{k-2} + \theta_1 \pi_{k-1} + \pi_k = 0$ so

$$\pi_k = -\theta_1 \pi_{k-1} - \theta_2 \pi_{k-2}$$

This is referred to as a <u>difference equation</u>

This model is invertible if

$$\lim_{k\to\infty} \pi_k = 0 .$$

ARMA (p,q)

$$x_t = \alpha + \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \cdots + \varphi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}$$

assuming stationarity,

$$E[x_t] = \alpha + \varphi_1 E[x_{t-1}] + \varphi_2 E[x_{t-2}] + \cdots + \varphi_p E[x_{t-p}] + 0$$

$$\mu = \alpha + \varphi_1 \mu + \varphi_2 \mu + \cdots \varphi_p \mu$$

$$\mu = \frac{\alpha}{1 - \varphi_1 - \varphi_2 - \cdots - \varphi_p}$$

As usual, we can alway write

$$y_t = x_t - \mu$$

$$y_t = \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}$$

Using backshift notation,

$$\varphi(B)\, y_t = \theta(B)\, w_t .$$

Given an ARMA(p,q) model, we can write it in MA(∞) form:

That is we seek $\psi(B)$ such that

$$y_t = \psi(B) w_t .$$

As before, substituting, we get

$$\varphi(B)\psi(B) w_t = \theta(B) w_t$$

So, I need to match coefficients of

$$\varphi(z)\psi(z) = \theta(z)$$

We can also write the ARMA(p,q) model in AR(∞) form:

That is $\pi(B) x_t = w_t$

Do this by substituting

$$\varphi(B) x_t = \theta(B)\pi(B) x_t$$

and match coefficients of

$$\theta(z)\pi(z) = \varphi(z) .$$

Parameter redundancy

Consider the model

$$y_t = 0.5 y_{t-1} + w_t - 0.5 w_{t-1}$$

In operator notation,

$$(1-0.5B) y_t = (1-0.5B) w_t$$

this model is equivalent to

$$y_t = w_t$$

In general, given the model

$$\varphi(B) y_t = \theta(B) w_t$$

the model

$$\eta(B)\varphi(B) y_t = \eta(B)\theta(B) w_t$$

is identical to it.

For identifiability, we will always require that $\varphi(B)$ & $\theta(B)$ have no common factors.

So, we usually impose the following 3 conditions:

(1) (No parameter redundancy) $\varphi(z)$ & $\theta(z)$ have no common factors

(2) (causal) $\{y_t\}$ can be written in form

$$y_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t$$

where $\psi_0 = 1$ & $\sum_{j=0}^{\infty} |\psi_j| < \infty$

(3) (invertible) $\{y_t\}$ can be written in form

$$w_t = \sum_{j=0}^{\infty} \pi_j y_{t-j} = \pi(B) y_t$$

where $\pi_0 = 1$ & $\sum_{j=0}^{\infty} |\pi_j| < \infty$.