Standard assumptions for ARMA(p,q) model

$$\varphi(B)\, y_t = \theta(B)\, w_t$$

(1) causal: $\{y_t\}$ ~~#~~ can be written in form

$$y_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B)\, w_t$$

where $\psi_0 = 1$ & $\sum_{j=0}^{\infty} |\psi_j| < \infty$

(2) Invertible: $\{y_t\}$ can be written in form

$$w_t = \sum_{j=0}^{\infty} \pi_j y_{t-j} = \pi(B)\, y_t$$

where $\pi_0 = 1$ & $\sum_{j=0}^{\infty} |\pi_j| < \infty$

(3) No parameter redundancy

$\varphi(z)$ & $\theta(z)$ have no common factors.

Example: consider the model

(*) $\qquad y_t = w_t \quad , \quad w_t \sim WN(\sigma_w^2)$

Now suppose we try fitting an ARMA(1,1) model to data generated from (*).

(**) $\qquad (1 - \phi_1 B)\, y_t = (1 + \theta_1 B)\, w_t$

recall that (**) & (*) are identical

if $\varphi_1 = -\theta_1$. In practice I will typically find that the the hypothesis $\hat{\varphi} = -\hat{\theta}_1$ is not rejected.

__Theorem__: An ARMA(p,q) model is causal iff all roots of $\varphi(z)$ lie outside the unit circle (in the complex plane).

__Theorem__: An ARMA (p,q) model is invertible iff all roots of $\theta(z)$ lie outside the unit circle (in the complex plane).

__Example__. $y_t = \varphi_1 y_{t-1} + w_t$

$$\varphi(z) = 1 - \varphi_1 z$$

$$1 - \varphi_1 z = 0 \quad \text{iff} \quad z = \frac{1}{\varphi_1}$$

This is outside the unit circle iff $|\varphi_1| < 1$.

__Example__: $y_t = 2y_{t-1} + 3y_{t-2} + w_t$

$$\varphi(z) = 1 - 2z - 3z^2 = 0$$

$$\| $$

$$(1-3z)(1+z)$$

has roots $z = 1/3, -1$

Neither are outside unit circle, so not causal.

__Example__:

$$y_t = w_t - \tfrac{1}{4} w_{t-2}$$

$$\theta(z) = 1 - \tfrac{1}{4} z^2$$

$$= (1 - \tfrac{1}{2} z)(1 + \tfrac{1}{2} z)$$

has roots

$$z = \pm 2$$

Both roots are outside unit circle, so invertible.

__Example__:

$$y_t = 0.4\, y_{t-1} + 0.45\, y_{t-2} + w_t + w_{t-1} + 0.25\, w_{t-2}$$

$$\varphi(z) = 1 - 0.4z - 0.45z^2 = \cancel{(1+0.5z)}(1-0.9z)$$

$$\theta(z) = 1 + z + 0.25z^2 = (1+0.5z)^{\cancel{2}}$$

there is parameter redundancy. after eliminating the redundancy, the model is

$$y_t - 0.9\, y_{t-1} = w_t + 0.5\, w_{t-1}$$

is both invertible & causal.

We learned previously how to write an ARMA (p,q) model in AR(∞) or MA(∞) using coefficient matching.

Can also do using R functions

ARMAtoAR( )

ARMAtoMA( )

3.3 Difference equations

A homogeneous difference equation of order p is of form:

$$u_n = \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + \cdots \alpha_p u_{n-p}$$

Using lag operator notation, we can write this as

$$\alpha(B) u_n = 0$$

where $\alpha(z) = 1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_p z^p$

is referred to as the characteristic polynomial.

Given initial conditions $u_0, u_1 \cdots u_{p-1}$, we can easily solve for $u_p, u_{p+1}, u_{p+2}, \ldots$

Theorem
It turns out that $\{u_i\}$ has the property

$$\lim_{i \to \infty} u_i = 0$$

iff the roots of the characteristic polynomial all outside the unit circle.

Remark: If we write an ARMA(p,q) model in AR(∞) form, the coefficients $\pi_1, \pi_2, \pi_3 \cdots$ form a difference equation of order q with characteristic polynomial $\theta(z)$ & initial conditions are determined by $\varphi_1, \varphi_2 \cdots \varphi_p$.

Analogously, if we try to write the model in MA(∞) form, the coefficients $\psi_1, \psi_2 \cdots$ form a difference equation of order p with characteristic polynomial $\varphi(z)$ & initial conditions determined by $\theta_1, \theta_2, \cdots \theta_q$.

Yule-Walker equations

Example: ACF of AR(2)

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + w_t .$$

Assume causal.

Multiplying through by y_{t-h} & taking $E[\cdot]$, we get

$$(*) \qquad E[y_{t-h} \cdot y_t] = \varphi_1 E[y_{t-h} \cdot y_{t-1}] + \varphi_2 E[y_{t-h} y_{t-2}] + E[y_{t+h} w_t]$$

Note that $E[y_{t-h} w_t] = \begin{cases} \sigma_w^2 & h=0 \\ 0 & h=1,2,3\ldots \end{cases}$

So $(*)$ implies

$$\left.\begin{array}{ll} (h=0) & \gamma_0 = \varphi_1 \gamma_1 + \varphi_2 \gamma_2 + \sigma_w^2 \\ (h=1) & \gamma_1 = \varphi_1 \gamma_0 + \varphi_2 \gamma_1 \\ (h=2) & \gamma_2 = \varphi_1 \gamma_1 + \varphi_2 \gamma_0 \end{array}\right\}$$

Yule-Walker equations.

If we know σ_w^2, 3 eq's in 3 unknowns.

Can solve for γ_0, γ_1 & γ_2

$$(h=3) \quad \gamma_3 = \varphi_1 \gamma_2 + \varphi_2 (\gamma_1)$$

$$(\cancel{h=4}): \quad \gamma_h = \varphi \gamma_{h-1} + \varphi_2 \gamma_{h-2} \qquad h = 3, 4, 5 \ldots$$

~~Recall, AR(1)~~ $\cancel{y_t = \phi y_{t-1} + w_t}$

$$\cancel{\gamma_h = \frac{\phi^h \sigma_w^2}{1-\phi^2}}$$

In practice it is easier to divide the Yule-Walker equations by γ_0

$$\rho_0 = 1$$

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1$$

$$= \phi_1 / 1 - \phi_2$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0$$

$$\rho_h = \phi_1 \rho_{h-1} + \phi_2 \rho_{h-2} \quad , h = 2, 3, 4, \ldots$$

So, $\gamma_h = \rho_h \cdot \gamma_0$

where $\gamma_0 = \dfrac{\sigma_w^2}{1 - \phi_1 \rho_1 - \phi_2 \rho_2}$

ACF of AR(p)

$$y_t = \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + w_t$$

Assume causal.

$$E[y_{t-h} y_t] = \varphi_1 E[y_{t-h} y_{t-1}] + \cdots + \varphi_p E[y_{t-h} y_{t-p}] + E[y_{t-h} w_t]$$

Note that $E[y_{t-h} w_t] = \begin{cases} \sigma_w^2 & h = 0 \\ 0 & h > 0 \end{cases}$

$$\gamma_{-h} = \gamma_h = \varphi_1 \gamma_{h-1} + \varphi_2 \gamma_{h-2} + \cdots \varphi_p \gamma_{h-p} \qquad h = 1, 2, 3 \cdots$$

is a difference equation with initial conditions

$$\left.\begin{aligned} \gamma_0 &= \varphi_1 \gamma_1 + \cdots + \varphi_p \gamma_p + \sigma_w^2 \\ \gamma_1 &= \varphi_1 \gamma_0 + \cdots + \varphi_p \gamma_{p-1} \\ &\vdots \\ \gamma_p &= \varphi_1 \gamma_{p-1} + \cdots \varphi_p \gamma_0 \end{aligned}\right\} \text{Yule-Walker equations}$$

Given σ_w^2, this is p equations in p unknowns, so I can solve for

$\gamma_0, \gamma_1 \cdots \gamma_{p-1}$

For next time read about PACF.