ACF of AR(2)

Assume causal

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$$

$$\gamma_h = \phi_1 \gamma_{h-1} + \phi_2 \gamma_{h-2} \qquad h = 1, 2, 3, \dots$$

difference equation.

Initial conditions given by solving Yule-Walker equations.

$$\left.\begin{aligned} \gamma_0 &= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_w^2 \\ \gamma_1 &= \phi_1 \gamma_0 + \phi_2 \gamma_1 \\ \gamma_2 &= \phi_1 \gamma_1 + \phi_2 \gamma_0 \end{aligned}\right\}$$

3 eq's in 3 unknown (assumes σ_w^2 is known).

In practice, this can be done slightly more easily

$$\rho_0 = 1$$

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_{-1}$$

$$= \frac{\phi_1}{1 - \phi_2}$$

Example:

$$y_t = 3/2\, y_{t-1} - 3/4\, y_{t-2} + w_t$$

$$\phi(z) = 1 - 3/2\, z + 3/4\, z^2$$

roots: $z = 1 \pm i/\sqrt{3}$

outside unit circle so causal.

$\rho_0 = 1$

$\rho_1 = \dfrac{\varphi_1}{1-\varphi_2} = \dfrac{3/2}{1+3/4} = 6/7$

$\rho_h = \varphi_1 \rho_{h-1} + \varphi_2 \rho_{h-2} \qquad h = 2, 3, \ldots$

Now simulate data from this model, compute sample ACF, theoretical ACF & plot.

<u>ACF of AR(p)</u>

$y_t = \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + w_t$

ACF is given by difference equation

$\gamma_h = \varphi_1 \gamma_{h-1} + \cdots + \varphi_p \gamma_{h-p} \qquad h = 1, 2, 3, \ldots$

with initial given by solving Yule-Walker equations (recognizing $\gamma_h = \gamma_{-h}$)

$$\text{p+1 eq's in p+1 unknowns} \left\{ \begin{aligned} \gamma_0 &= \varphi_1 \gamma_{-1} + \cdots + \varphi_p \gamma_{-p} + \sigma_w^2 \\ \gamma_1 &= \varphi_1 \gamma_0 + \cdots + \varphi_p \gamma_{1-p} \\ &\vdots \\ \gamma_p &= \varphi_1 \gamma_{p-1} + \cdots + \varphi_p \gamma_0 \end{aligned} \right.$$

ACF of MA(1)

$$y_t = w_t + \theta w_{t-1}$$

$$\gamma_0 = E[y_t y_t] = E[(w_t + \theta w_{t-1})(w_t + \theta w_{t-1})]$$

$$= \sigma_w^2 + \theta^2 \sigma_w = (1+\theta^2)\sigma_w^2$$

$$\gamma_1 = \gamma_{-1} = E[y_t \cdot y_{t-1}] = E[(w_t + \theta w_{t-1})(w_{t-1} + \theta w_{t-2})]$$

$$= \theta \sigma_w^2$$

$$\gamma_h = 0 \quad \text{for } h = \pm 2, \pm 3, \ldots$$

ACF of MA(q)

$$y_t = \sum_{j=0}^{q} \theta_j w_{t-j}$$

Assume $h > 0$

$$\gamma(-h) = \gamma(h) = E\left[y_{t-h} \cdot y_t \right]$$

$$= E\left[\left(\sum_{j=0}^{q} \theta_j w_{t-h-j} \right) \left(\sum_{k=0}^{q} \theta_k w_{t-k} \right) \right]$$

$$= \begin{cases} \sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} & 0 \le h \le q \\ 0 & \text{else} \end{cases}$$

ACF of ARMA(1,1)

$$y_t = \varphi y_{t-1} + w_t + \theta w_{t-1}$$

Assume causal. So, we can write

$$y_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

Notice that

$$E[y_t \cdot w_t] = E\left[\left(\sum_{j=0}^{\infty} \psi_j w_{t-j}\right) \cdot w_t\right]$$

$$= \sigma_w^2 \cdot \psi_0$$

$$E[y_t \cdot w_{t-1}] = E\left[\left(\sum_{j=0}^{\infty} \psi_j w_{t-j}\right) \cdot w_{t-1}\right]$$

$$= \sigma_w^2 \, \psi_1$$

$$\vdots$$

$$E[y_t \cdot w_{t-h}] = \begin{cases} \sigma_w^2 \psi_h & \text{for } h \geq 0. \\ 0 & \text{for } h < 0 \end{cases}$$

Now let's compute the ACF.

$$\gamma_{-h} = \gamma_h = E[y_t \cdot y_{t+h}]$$

$$= E[y_t \cdot (\varphi y_{t+h-1} + w_{t+h} + \theta w_{t+h-1})]$$

$$= \varphi E[y_t \cdot y_{t+h-1}] + E[y_t \cdot w_{t+h}] + \theta E[y_t \cdot w_{t+h-1}]$$

$$= \varphi E[y_t \cdot y_{t+h-1}] + E[y_t w_{t+h}] + \theta E[y_t w_{t+h-1}]$$

for $h = 2, 3, \ldots$

$$\gamma_h = \varphi \gamma_{h-1} + 0 + 0$$

$$\left.\begin{array}{lll} h=0 & \gamma_0 = \varphi \gamma_1 + \psi_0 \sigma_w^2 + \theta \psi_1 \sigma_w^2 \\ h=1 & \gamma_1 = \varphi \gamma_0 + 0 + \theta \psi_0 \sigma_w^2 \end{array}\right\} \text{Yule-walker}$$

Note that this is the same as for the AR(1) but with the different initial conditions (given by solving the Yule-Walker equations).

ARMA(p,q)

$$y_t = \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + w_t + \theta_1 w_{t-1} + \cdots \theta_q w_{t-q}$$

$$\gamma_h = E\{y_t \cdot y_{t+h}\}$$
$$= \cdots$$

As before, we get

$$\gamma_h = \varphi_1 \gamma_{h-1} + \cdots + \varphi_p \gamma_{h-p} \qquad \text{for } h > q$$

With initial conditions given by solving the Yule-Walker equations:

$$\gamma_h = \varphi_1 \gamma_{h-1} + \cdots + \varphi_p \gamma_{h-p} + \sigma_w^2 \sum_{i=0}^{q} \theta_i \psi_{h-i}$$

for $h = 0, 1, 2 \cdots p$

PACF

given $\{y_t\}$ (assume $\mu_y = 0$).

Consider the population regression

$$y_t = \varphi_{i1} y_{t-1} + \cdots + \varphi_{ii} y_{t-i} + \varepsilon_t$$

where

$$\begin{pmatrix} \varphi_{i1} \\ \vdots \\ \varphi_{ii} \end{pmatrix} \stackrel{def}{=} \left[E(\underline{X}'\underline{X}) \right]^{-1} E(\underline{X}'\underline{Y})$$

with $\underline{X} = (y_{t-1}, \cdots, y_{t-i})$

$\underline{Y} = y_t$

this is also called "linear projection"

We refer to φ_{ii} as the i^{th} PACF.

We can think of ρ_i as being motivated by the univariate regression $y_t = \rho_i y_{t-i} + \varepsilon_t$

φ_{ii} is motivated by the multivariate regression

$$y_t = \varphi_{i1} y_{t-1} + \cdots + \varphi_{ii} y_{t-i} + \varepsilon_t$$

$$\begin{array}{llll} \boxed{\varphi_{11}} & & & \\ \varphi_{21} & \boxed{\varphi_{22}} & & \\ \varphi_{31} & \varphi_{32} & \boxed{\varphi_{33}} & \\ \varphi_{41} & \varphi_{42} & \varphi_{43} & \boxed{\varphi_{44}} \end{array}$$

these are the ones I care about

$$y_t = \varphi_{11} y_{t-1} + \varepsilon_t$$

$$y_t = \varphi_{21} y_{t-1} + \varphi_{22} y_{t-2} + \varepsilon_t$$

$$y_t = \varphi_{31} y_{t-1} + \varphi_{32} y_{t-2} + \varphi_{33} y_{t-3} + \varepsilon_t$$

To estimate φ_{ii} given $\{y_t\}$

do the regression

$$y_t = \hat{\varphi}_{i1} y_{t-1} + \cdots + \hat{\varphi}_{ii} y_{t-i} + \varepsilon_t$$

this is the ith Partial autocorrelation.

Example:

given the AR(2) model,

$$x_t = 3/2\, x_{t-1} - 3/4\, x_{t-2} + w_t$$

(1) is it causal?

(2) Compute & plot theoretical ACF & PACF

(3) Simulate 1000 observations & plot sample ACF & PACF

Summary

AR(p) : ACF decays exponentially

PACF cuts off after p lags

MA(q) : ACF cuts off after q lags

PACF decays exponentially

So, given data, behavior of ACF & PACF can be used in the process of model exploration.