Review

Example: ARMA(2,2)

$$y_t = \tfrac{5}{6} y_{t-1} - \tfrac{1}{6} y_{t-2} + w_t - 1.7 w_{t-1} + 0.72 w_{t-2}$$

$$\varphi(B) y_t = \theta(B) w_t$$

where $\varphi(z) = 1 - \tfrac{5}{6} z + \tfrac{1}{6} z^2$

$= (1 - z/3)(1 - z/2)$ $\quad |3|>1$ & $|2|>1$

roots at $z = 3, 2$. So it is causal

$\theta(z) = 1 - 1.7z + 0.72 z^2$

$= (1 - 0.9z)(1 - 0.8z)$

roots at $z = 1/0.9, 1/0.8$ so it is invertible.

- Get Wold decomposition (ie, $MA(\infty)$ form):

given $\varphi(B) y_t = \theta(B) w_t$

we seek $\psi(B)$ such that

$$y_t = \psi(B) w_t .$$

Substituting, we get

$$\varphi(B) \psi(B) w_t = \theta(B) w_t$$

So, we must have

$$\varphi(z) \psi(z) \equiv \theta(z) \quad \text{for all } z.$$

$$\left(1 - \tfrac{5}{6} z + \tfrac{1}{6} z^2\right)\left(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots\right) = 1 - 1.7z + 0.72 z^2$$

z: $\psi_1 - \tfrac{5}{6} = -1.7$, $\quad \psi_1 = \tfrac{5}{6} - 1.7 = -0.8667$

z^2: $\psi_2 - \tfrac{5}{6}\psi_1 + \tfrac{1}{6} = 0.72$ $\quad \psi_2 = \tfrac{5}{6}\psi_1 - \tfrac{1}{6} + 0.72$

$= -0.16889$

z^3: $\psi_3 - 5/6\,\psi_2 + 1/6\,\psi_1 = 0$ $\qquad \psi_3 = 5/6\,\psi_2 - 1/6\,\psi_1$
$\qquad = 0.0037$

$\vdots$

z^k: $\psi_k - 5/6\,\psi_{k-1} + 1/6\,\psi_{k-2} = 0$

Note: This is a difference equation of order 2 depending φ with initial conditions that depend on θ.

We could also find AR(∞) form...
(exercise)

Now, let's compute the ACF.

Recall, $\gamma_h = \varphi_1 \gamma_{h-1} + \varphi_2 \gamma_{h-2} + \sigma_w^2 \sum_{i=0}^{q} \theta_i \psi_{i-h}$

check your notes. May have flipped sign previously.

Yule Walker equations

h=0

$\gamma_0 = \varphi_1 \gamma_1 + \varphi_2 \gamma_2 + \sigma_w^2 \left[1 + \theta_1 \psi_1 + \theta_2 \psi_2\right]$

recall $\gamma_1 = \gamma_{-1}$
$\gamma_2 = \gamma_{-2}$

h=1

$\gamma_1 = \varphi_1 \gamma_0 + \varphi_2 \gamma_1 + \sigma_w^2 \left[1 \cdot 0 + \theta_1 \psi_0 + \theta_2 \psi_1\right]$

recall $\psi_j = 0$ for $j < 0$

h=2 $\gamma_2 = \varphi_1 \gamma_1 + \varphi_2 \gamma_0 + \sigma_w^2 \left[1 \cdot 0 + \theta_1 \cdot 0 + \theta_2 \cdot 1\right]$

since $\theta_0 = 1$ $\qquad$ since $\psi_0 = 1$

$h>2 \quad \gamma_h = \varphi_1 \gamma_{h-1} + \varphi_2 \gamma_{h-2}$

Notice that this is a difference equation of order 2 ~~with~~ that depends only on φ with initial conditions that depend on θ.

get the initial conditions by solving Yule-Walker equations for γ_0, γ_1 & γ_2.

Now, let's compute the PACF

To get φ_{hh}, consider the population regression

$$y_t = \varphi_{h1} y_{t-1} + \cdots + \varphi_{hh} y_{t-h} + \varepsilon_t$$

$$X = \begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-h} \end{bmatrix} \qquad Y = y_t$$

$$\begin{bmatrix} \varphi_{h1} \\ \vdots \\ \varphi_{hh} \end{bmatrix} = \left[E(X'X)\right]^{-1} E(X'Y)$$

recall: φ_{hh} is the only coefficient I care about.

$$= \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{h-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & & \\ & \ddots & & \ddots & \\ \gamma_{h-1} & & & & \gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_h \end{bmatrix}$$