Econ 522 Midterm, part 1 (Spring 2020)

Do the following problems using pencil and paper. Scan and submit on polylearn.

1. Consider the time series
$$y_t = \delta + y_{t-1} + w_t, \qquad w_t \sim WN(\sigma_W^2)$$
with initial condition $y_0 = 0$. Is $\{y_t\}$ stationary? What is the mean function, $\mu_y(t)$? What is the autocovariance function, $\gamma_y(s,t)$?

Solution:

Write
$$y_t = t\delta + \sum_{i=1}^{t} w_i.$$
Then,
$$\mu_y(t) = E(y_t) = t\delta$$
and
$$\gamma_y(s,t) = \text{cov}(y_s, y_t) = E\left(\sum_{i=1}^{s} w_i\right)\left(\sum_{j=1}^{t} w_j\right) = \min\{s,t\}\sigma_w^2.$$
Not stationary since μ_y is a function of t.

2. Consider the time series
$$\begin{aligned} x_t &= \phi x_{t-1} + w_t \\ y_t &= \alpha + w_t + u_t + \theta u_{t-1} \end{aligned}$$
where $|\phi| < 1$ and $\{w_t\}$ and $\{u_t\}$ are independent white noise processes with variance σ_w^2 and σ_u^2 respectively.

(a) What is $\gamma_x(h)$, the autocovariance function of $\{x_t\}$?

(b) What is $\gamma_y(h)$, the autocovariance function of $\{y_t\}$?

(c) What is $\gamma_{xy}(h)$, the crosscovariance function of $\{x_t\}$ and $\{y_t\}$?

Solution:

(a) $\gamma_x = \phi^h \sigma_w^2/(1-\phi^2)$.

(b) $\gamma_y = \begin{cases} \sigma_w^2 + (1+\theta^2)\sigma_u^2 & h = 0 \\ \theta\sigma_u^2 & h = \pm 1 \\ 0 & \text{else.} \end{cases}$

(c) For $h \geq 0$, $\gamma_{xy}(h) = \mathrm{E}[x_{t+h} y_t] = \mathrm{cov}\left(\sum_{i=0}^{\infty} \phi^i w_{t+h-i}, \alpha + w_t + u_t + \theta u_{t-1}\right) = \phi^h \sigma_w^2$.
For $h < 0$, $\gamma_{xy}(h) = 0$.

3. Consider the model

$$y_t = \alpha - 1.7 y_{t-1} + 0.6 y_{t-2} + w_t + 0.4 w_{t-1},$$

where w_t is white noise with variance σ_w^2. Is this model causal? Is it invertible? (Why?)

Solution:

The AR polynomial, $\phi(z) = 1 + 1.7z - 0.6z^2$, has roots $z = (10/3, -0.5)$. One is inside and one is outside the unit circle, so the model is noncasual.

The MA polynomial, $\theta(z) = 1 + 0.4z$, has root $z = -2.5$, which is outside the unit circle, so invertible.

4. Consider the model

$$y_t = 0.3 y_{t-1} + w_t + 1.5 w_{t-1},$$

where w_t is white noise with variance σ^2.

Write this model in form $y_t = \sum_{i=0}^{\infty} \psi_i w_{t-i}$. (Show how to compute ψ_i, $i = 1, 2, \ldots$, in terms of the model coefficients.)

Solution:

(a) $\phi = 0.3$, so $|\phi| < 1$ and therefore causal. $\theta = 1.5$ so $|\theta| > 1$ and therefore not invertible.

(b) Given

$$\phi(B) y_t = \theta(B) w_t \tag{1}$$

we search $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$ such that

$$y_t = \psi(B) w_t. \tag{2}$$

Substituting (2) into (1), we get

$$\phi(B)\psi(B) w_t = \theta(B) w_t,$$

which implies

$$\phi(z)\psi(z) = \theta(z).$$

Writing this out explicitly, we get

$$(1 - 0.3z)(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots) = 1 + 1.5z.$$

Matching coefficients we get,

$$\begin{aligned} \psi_1 - 0.3 &= 1.5 & \psi_1 &= 1.8 \\ \psi_2 - 0.3\psi_1 &= 0 & \psi_2 &= 0.3\psi_1 = 0.54 \\ \psi_3 - 0.3\psi_2 &= 0 & \psi_3 &= 0.3\psi_2 = 0.162 \\ \psi_4 - 0.3\psi_3 &= 0 & \psi_4 &= 0.3\psi_3 = 0.0486 \\ &\vdots \\ \psi_n - 0.3\psi_{n-1} &= 0 \\ &\vdots \end{aligned}$$