Math boot camp

Linear algebra

Textbook:

Mathematics for economists

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Chapter 6

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

Examples

$$y = 3x + 4$$

$$x + 3y + 2z = 4$$

$$x_1 + 3x_2 + 2x_3 = 4$$

Example 1:

predtax earnings = 100 000

5% state tax = S

40% Ferderal tax = F

10% aftertax donation = C

$$C = 0.10(100\,000 - S - F)$$
$$S = 0.05(100\,000 - C)$$
$$F = 0.40(100\,000 - C - S)$$

or,

$$C + 0.10S + 0.10F = 10\,000$$
$$+0.05C + S = 5\,000$$
$$0.40C + 0.210S + F = 40\,000$$

$$\begin{bmatrix} 1 & 0.10 & 0.10 \\ 0.05 & 1 & 0 \\ 0.40 & 0.40 & 1 \end{bmatrix} \begin{bmatrix} C \\ S \\ F \end{bmatrix} = \begin{bmatrix} 10\,000 \\ 5\,000 \\ 40\,000 \end{bmatrix}$$

solve:

$C = 5956$

$S = 4702$

$F = 35737$

Example: Markov Model of Employment

week t week $t+1$

unemployed $\xrightarrow{p}$ employed

unemployed $\xrightarrow{1-p}$ unemployed

employed $\xrightarrow{q}$ employed

employed $\xrightarrow{1-q}$ unemployed

states: unemployed $= S_1$

employed $= S_2$

Let x_t = # employed

y_t = # unemployed

$$x_{t+1} = q x_t + p y_t$$

$$y_{t+1} = (1-q) x_t + (1-p) y_t$$

Assume $x_t + y_t = x_{t+1} + y_{t+1} = 1$

Is there a "steady state" level of employment,

$x_t = x_{t+1}$?   or $(1-q)x - py = 0$

$x = qx + py$ $(1-q)x + py = 0$

$y = (1-q)x + (1-p)y$ $x + y = 1$

$$\begin{bmatrix} q-1 & p \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x = \frac{p}{1+p-q}$$

$$y = \frac{1-q}{1+p-q}$$

Example : Investments

S states

N assets

W_0 = Wealth at $t=0$

V_i = Price of asset i at $t=0$

R_{si} = Return $(s=1 \dots S,\ i=1,\dots,N)$

Budget constraint:

$$n_1 V_1 + \cdots + n_N V_N = W_0$$

where $n_i \cdots$ is # of shares of asset i purchased

Let $x_i = \dfrac{n_i \cdot V_i}{W_0}$ = fraction of wealth invested in asset i.

$$x_1 + x_2 + \cdots x_N = 1$$

Can I find a Riskless portfolio?

portfolio value in state s

$$R_1 = R_{11} x_1 + R_{12} x_2 + \cdots + R_{1N} x_N$$

$$\vdots$$

$$R_S = R_{S1} x_1 + R_{S2} x_2 + \cdots + R_{SN} x_N$$

Can I find $(x_1 \cdots x_N)$ such that $R_1 = R_2 = \cdots = R_S$

Example:

$R_{11} = 1 \qquad R_{12} = 3$

$R_{21} = 2 \qquad R_{22} = 2$

$R_{31} = 3 \qquad R_{32} = 1$

$(R_1 = R_2) \qquad 1x_1 + 3x_2 = 2x_1 + 2x_2$

$(R_2 = R_3) \qquad 2x_1 + 2x_2 = 3x_1 + x_2$

$x_1 + x_2 = 1$

$-1x_1 + x_2 = 0$

$-1x_1 + x_2 = 0$

$x_1 + x_2 = 1$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

solve: $x_1 = x_2 = 1/2$

Chapter 7

Linear system of equations, m equations & n unknowns is of form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

a_{ij} are coefficients

x_j are unknowns $\qquad (i = 1, \ldots, m;\ j = 1 \ldots n)$

Does a solution exist?

Is it unique?

How to solve?

Solution by substitution

Example:

$$x - 0.4y - 0.3z = 130$$
$$-0.2x + 0.88y - 0.14z = 74$$
$$-0.5x - 0.2y + 0.95z = 95$$

solve eq 1 for x: $\quad x = 130 + 0.4y + 0.3z \qquad (**)$

plug into eq's 2 & 3 $\quad -0.2(130 + 0.4y + 0.3z) + 0.88y - 0.14z = 74$

$$-0.5(130 + 0.4y + 0.3z) - 0.2y + 0.95z = 95$$

collect terms

$$0.8y - 0.2z = 100$$
$$-0.4y + 0.8z = 160$$

solve for y: $.8y = 100 + 0.2z$

(*) $y = 125 + 0.25z$

plug into eq (3) $-0.4(125 + 0.25z) + 0.8z = 160$

collect terms: $0.7z = 210$

$\boxed{z = 300}$

plug into (*) $y = 125 + 0.25 \cdot 300$

$= \boxed{200}$

plug into (**) $x = 130 + 0.4(200) + 0.3(300)$

$= 130 + 80 + 90$

$= \boxed{300}$

ELimination

$$x - 2y = 8$$
$$3x + y = 3$$

$-3 \times R_1 + R_2$ $\quad + 7y = -21$

$$\boxed{y = -3}$$

plug into eq (1) : $\quad x + 6 = 8$

$$\boxed{x = 2}$$

Now, go back to previous example:

$$x - 0.4y - 0.3z = 130$$
$$-0.2x + 0.28y - 0.14z = 74$$
$$-0.5x - 0.2y + 0.95z = 95$$

(1) $\quad x - 0.4y - 0.3z = 130$

$+0.2$ (1) + (2) $\quad 0.8y - 0.2z = 100$

0.5 (1) + (3) $\quad -0.4y + 0.8z = 160$

(1) $\quad x - 0.4y - 0.3z = 130$

(2) $\quad 0.8y - 0.2z = 100$

$\frac{1}{2}$(2) + (3) $\quad 0.7z = 210$

$$z = \frac{210}{0.7} = 300$$

plug into (2) $\quad 0.8y - 0.2(300) = 100$

$$y = 200$$

plug into (1) $x - 0.4(200) - 0.3(300) = 130$

$x = 300$

Gauss - Jordan elimination

Instead of doing back substitution, do as follows:

$$
\begin{aligned}
x - 0.4y - 0.3z &= 130 \\
0.8y - 0.2z &= 100 \\
0.7z &= 210
\end{aligned}
$$

$$
\begin{aligned}
x - 0.4y - 0.3z &= 130 \\
\tfrac{1}{0.8}(2) \qquad y - 0.25z &= 125 \\
\tfrac{1}{0.7}(3) \qquad \boxed{z = 300}
\end{aligned}
$$

$$
\begin{aligned}
(1) + .30(3) \qquad x - 0.4y &= 220 \\
(2) + 0.25(3) \qquad \boxed{y = 200} \\
z &= 300
\end{aligned}
$$

$(1) + 0.4(2)$ $\boxed{x = 300}$

We can write a system of linear equations using matrices, as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$Ax = b$$

The augmented matrix

$$\hat{A} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

The following elementary row operations leave the system of equations unchanged (same solution):

(1) interchange 2 rows

(2) add a multiple of any row to a row

(3) multiply each element of a row by a nonzero constant.

$$x - 0.4y - 0.3z = 130$$
$$-0.2x + 0.88y - 0.14z = 74$$
$$-0.5x - 0.2y + 0.95z = 95$$

$$\left[\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ -0.2 & 0.88 & -0.14 & 74 \\ -0.5 & -0.2 & 0.95 & 95 \end{array}\right]$$

$$\begin{array}{l} \\ 0.2(1)+(2) \\ 0.5(1)+(3) \end{array} \left[\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 0.8 & -0.2 & 100 \\ 0 & -0.4 & 0.8 & 160 \end{array}\right]$$

$$\begin{array}{l} (1) \\ (2) \\ \frac{1}{2}(2)+(3) \end{array} \left[\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 0.8 & -0.2 & 100 \\ 0 & 0 & 0.7 & 210 \end{array}\right]$$

$$\begin{array}{l} (1)+0.3(3) \\ (2)+0.2(3) \\ (3)/0.7 \end{array} \left[\begin{array}{ccc|c} 1 & -0.4 & 0 & 220 \\ 0 & 0.8 & 0 & 160 \\ 0 & 0 & 1 & 300 \end{array}\right]$$

$$\begin{array}{l} R_1+0.4(2) \\ R_2/0.8 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 200 \\ 0 & 0 & 1 & 300 \end{array}\right]$$

$$\boxed{\begin{aligned} x &= 300 \\ y &= 200 \\ z &= 300 \end{aligned}}$$

Row Echelon form

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Each row has more leading zeros than the row above.

Reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Row echelon form but at most one non zero element in each row.

$$\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$ neither

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$ row echelon

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ reduced row echelon

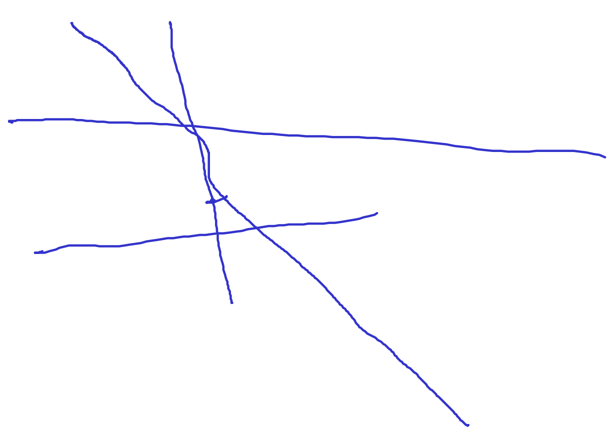
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ reduced row echelon

A system of equations can have

- 1 solution
- 0 solutions
- ∞ solutions

Example:

$$x + 3y = 1$$
$$x + y = 2$$



$$y = -x/3 + 1/3$$
$$y = -x + 2$$

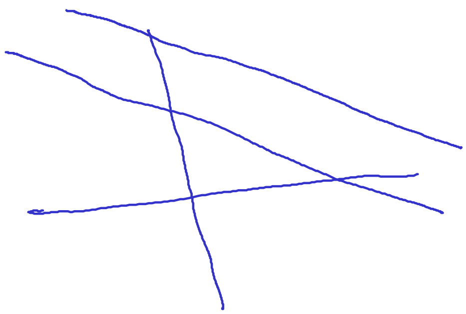
$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

(2) − (1) $$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -2 & 1 \end{array}\right]$$

(1) − 3(2)
−0.5(2) $$\left[\begin{array}{cc|c} 1 & 0 & 4.5 \\ 0 & 1 & -0.5 \end{array}\right]$$

Example:

$$x + 2y = 3$$
$$3x + 6y = 8$$



$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 6 & 8 \end{array}\right]$$

$$\begin{array}{l} (1) \\ (2) - 3\cdot(1) \end{array} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & -1 \end{array}\right]$$

a row of zeros with non-zero RHS indicates no solution.

Example:

$$x + 2y = 3$$
$$3x + 6y = 9$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 6 & 9 \end{array}\right]$$

$$\begin{array}{l} (1) \\ (2) - 3(1) \end{array} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

~~A reduced row echelon form with a row of zeros & RHS of that row is also zero indicates an infinite # of solutions.~~

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$ 1 solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 0 & 0 & 0 & 2 \end{array}\right]$$ 0 solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ 1 solution

$$\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{array}\right]$$ No solution

Consider a matrix in row echelon form

$$\left[\begin{array}{ccccc|c} * & w & w & w & w & w \\ 0 & * & w & w & w & w \\ 0 & 0 & 0 & * & w & w \end{array}\right]$$

*'s are called pivot

w's may be zero or non-zero.

If column j has a pivot, then x_j is a "basic variable"

If not, it is a "free variable."

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 0 & 5 & 4 \end{array}\right]$$

$\uparrow$ pivot $\uparrow$

x_1 & x_3 are basic

x_2 is free

$$1x_1 + \quad 3x_3 = 2$$
$$5x_3 = 4$$

There are an infinite # of solutions. (use any value for x_2 & solve for x_1 & x_3)

example

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

x_1 & x_3 are basic
x_2 is free

$$x_1 + 3x_3 = 4 - 2x_2$$
$$x_3 = 5$$

given any value for x_2, can solve for x_1 & x_3

Given an augmented matrix in row echelon form:

- If there is a row of zero with non-zero RHS: no solution
- If there are no free variables (i.e., every column is a pivot), then the basic variables are uniquely defined.
- If there are free variables, then there are infinitely many solutions if there are any solutions.

Rank

The number of nonzero rows in row echelon form.

A = coefficient matrix
$\hat{A}$ = augmented matrix

$\text{Rank } A \leq \text{Rank } \hat{A}$

$\text{Rank } A \leq \#$ of rows in A
$\text{Rank } A \leq \#$ of columns in A

A system of linear equations has a solution iff $\text{Rank } \hat{A} = \text{Rank } A$

$\text{Rank } A = \#$ rows
then at least one solution

$\text{Rank } A = \#$ rows
more columns than rows
: infinite # of solutions

$\text{Rank } A = \#$ of rows $= \#$ columns
: exactly one solution