Math Boot camp
(linear algebra)
Day 2

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

i ↑ row, j ↑ col

$M \times N$

↑ rows ↖ column

$B = (b_{ij})$ $K \times L$

$C = (c_{ij})$

$A + B = C$ $c_{ij} = a_{ij} + b_{ij}$

$K = M$
$L = N$ (shapes are same)

$A - B = C$ $c_{ij} = a_{ij} - b_{ij}$

$AB = C$ $c_{ij} = \sum_{k=1}^{K} a_{ik} b_{kj}$

$N = K$

Not commutative.
In general $AB \neq BA$

Associative

$(A+B)+C = A+(B+C)$

$(AB)C = A(BC)$

Addition is commutative

$$A + B = B + A$$

Distributive

$$A \cdot (B + C) = AB + AC$$

Transpose

$$A^T = (a_{ji})$$

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(AB)^T = B^T A^T$$

Recall system of linear equations

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Special Matrices

Square MATRIX

column MATRIX

row MATRIX

Diagonal MATRIX $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Zero MATRIX

Identity MATRIX $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$A + O = A$$

$$A \cdot O = O$$

$$A \cdot I = A$$

Upper Triangular $\begin{pmatrix} 3 & 4 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Lower Triangular $\begin{pmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 5 & 6 \end{pmatrix}$

symmetric $A = A^T$ $\begin{pmatrix} 3 & 1 & 6 \\ 1 & 4 & 5 \\ 6 & 5 & 2 \end{pmatrix}$

IDEMPOTENT

$$A \cdot A = A$$

$$\begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}$$

PERMUTATION

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$$

Nonsingular:

$n \times n$ square MATRIX

- RANK $= n$

  (i.e., no rows of all zeros in row echelon form).

- or equivalently, there is a pivot element in each column.

So, if A is nonsingular, it can be written in row echelon form as

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

Recall. If A is nonsingular, then $Ax=b$ has a unique solution.

Inverses

B is a right inverse of A if

$$A \cdot B = I$$

B is a left inverse of A if

$$B \cdot A = I$$

If A has a right inverse B & a left inverse C, then $B=C$ and we say that A has an inverse, A^{-1}.

For a square matrix, if it has a right or a left inverse, then it is invertible.

A square matrix is invertible iff it is nonsingular.

Recall, if A is nonsingular,

$$Ax = b$$

always has a unique solution

Recall: the inverse of A satisfies

$$A \cdot A^{-1} = I$$

$$\begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has a right inverse but no left inverse

It does not have a left inverse.

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 5 & 0 \end{bmatrix}$$

has a left inverse but no right inverse.

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$A/B = A \cdot B^{-1}$$

$$A \backslash B = A^{-1} B$$

Determinants

Let A be a 1×1 matrix.

$$A = (a)$$

$$|A| = a$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $|A| = ad - bc$

Suppose $A = (a_{ij})$

define A_{ij} is A with i^{th} row & j^{th} column deleted.

If A is $n \times n$,

$$|A| = \sum_{j=1}^{n} a_{ij} |A_{ij}| (-1)^{i+j}$$

$$A = \begin{pmatrix} 3 & 2 & 7 \\ 4 & 5 & 2 \\ 1 & 6 & 1 \end{pmatrix}$$

$$|A| = 3 \cdot \begin{vmatrix} 5 & 2 \\ 6 & 1 \end{vmatrix} \cdot (-1)^{1+1} + 2 \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} \cdot (-1)^{1+2} + 7 \underbrace{\begin{vmatrix} 4 & 5 \\ 1 & 6 \end{vmatrix}} (-1)^{1+3}$$

$$= 3 \cdot (5-12) \cdot 1 + 2 \cdot (4-2) \cdot -1 + 7 \cdot (24-5) \cdot 1$$

$$= -21 - 4 + 133$$

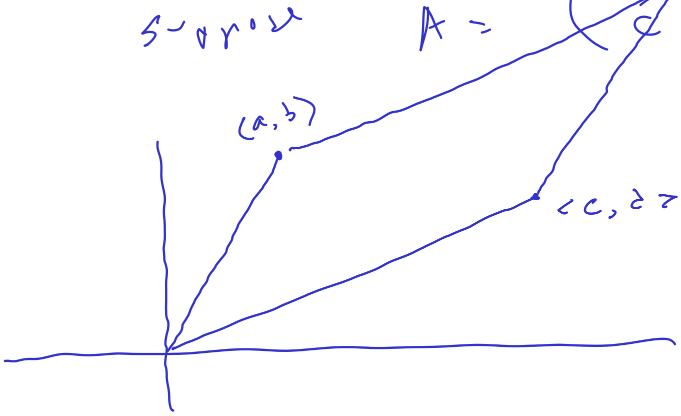
$$= \boxed{108}$$

$|A_{13}| = (1,3)^{th}$ minor of A

$|A_{13}| \cdot (-1)^{1+3} = (1,3)^{th}$ cofactor of A

In two dimensions

suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.



$|A|$ = Area of parallelogram

Recall elementary row operations

-

-

-

Doing any elementary row operation leaves determinant unchanged, except if I interchange rows i & j the determinant is multiplied by $(-1)^{i+j}$

So if I put a square matrix in row echelon form, the determinant has same magnitude.

If A is Triangular,

$|A|$ = product of diagonal elements.

Example:

$$A = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$$

$$|A| = 3 \cdot 2 - (0 \cdot 4) = 6$$

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 6 & 2 \end{pmatrix}$$

$$|A| = 3 \cdot \begin{vmatrix} 1 & 0 \\ 6 & 2 \end{vmatrix}$$

$$+ \; 0 \cdot \begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} \cdot -1 \qquad = 3 \cdot 1 \cdot 2$$

$$+ \; 0 \cdot \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix}$$

$|A| = 0$ iff A is singular.

Example

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 0 & 2 \\ 1 & 6 & 7 \end{pmatrix}$$

$3 \cdot R_2 - 2R_1$

$3R_3 - R_1$

$$\begin{pmatrix} 3 & 4 & 7 \\ 0 & -8 & -8 \\ 0 & 14 & 14 \end{pmatrix}$$

$8 \cdot R_3 + 14 \cdot R_2$ $\begin{pmatrix} 3 & 4 & 7 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{pmatrix}$

A is singular

$|A| = 0$

A has no inverse

$Ax = b$

~~has 0 solution if $b \neq 0$~~

$\rightarrow$ has either 0 solutions or ∞ solutions.

$|A^T| = |A|$

$|A^{-1}| = 1/|A|$

$|AB| = |A||B|$

Linear dependence & spanning sets

Given column vectors $v_1, v_2 \cdots v_k$,

the set $V = \mathcal{L}(v_1, v_2 \cdots v_k)$

is referred to as the set <u>spanned by</u>

$v_1 \cdots v_k$. is defined as

$$\{ c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \text{ for } c_1 \cdots c_k \in \mathbb{R} \}.$$

$c_1 v_1 + \cdots + c_k v_k$ is referred to as a

<u>linear combination</u> of $v_1 \cdots v_k$.

$v_1, \cdots v_k$ are said to be <u>linearly</u>

<u>independent</u> if

$$c_1 v_1 + \cdots + c_k v_k = 0$$

only for $c_1 = \cdots = c_k = 0$.

Or, equivalently, we can not write

$$v_k = c_1 v_1 + \cdots c_{k-1} v_{k-1} + \cdots c_{k+1} v_{k+1} \cdots + c_k v_k$$

To find $c_1 \cdots c_K$ such that

$c_1 v_1 + \cdots + c_K V_K = 0$, solve

$$\begin{bmatrix} v_1 & v_2 & & V_K \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_K \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

There are an infinite # of solutions if $\text{Rank}([V_1 \cdots V_K]) < K$.

So, if $\text{Rank}([V_1 \cdots V_K]) < K$, $\{v_1 \cdots v_K\}$ are linearly dependent.

Let $V = \mathcal{L}(v_1 \cdots V_K)$.

$\{v_1 \cdots v_K\}$ is referred to as a "basis" for V.

If $\{v_1 \cdots v_K\}$ are linearly dependent, then I don't need all of them to span V.

The smallest set of $\{v_k\}$ that spans V is referred to a an <u>efficient basis</u>.

The elements of an efficient basis are always linearly independent.

Example:

$$[v_1 \; v_2 \; v_3] = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 1 & 3 \\ 3 & 2 & 8 \\ 4 & 4 & 12 \\ 5 & 1 & 11 \end{bmatrix}$$

put in row echelon form:

$$\begin{array}{l} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \\ R_4 - 2R_1 \\ 2R_5 - 5R_1 \end{array} \begin{bmatrix} 2 & 3 & 7 \\ 0 & -1 & -1 \\ 0 & -5 & -5 \\ 0 & -2 & -2 \\ 0 & -13 & -13 \end{bmatrix}$$

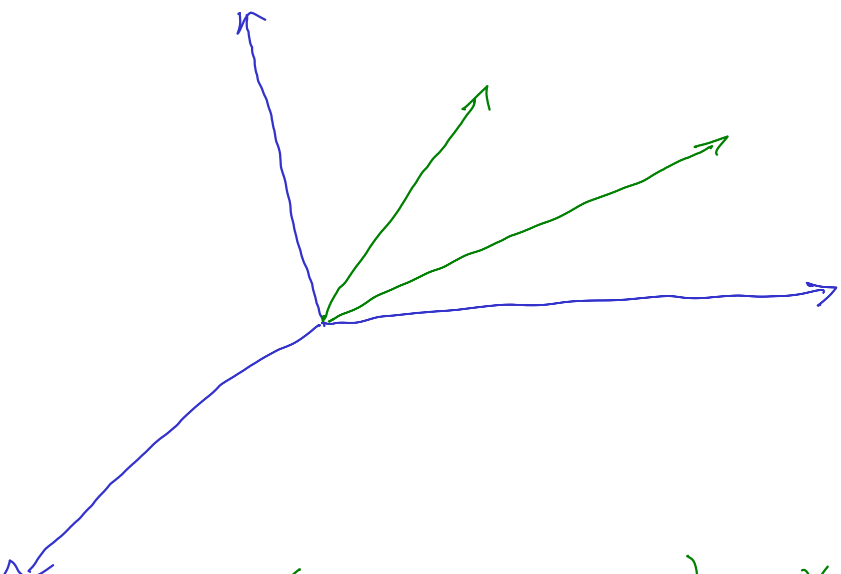
$$\begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 2R_2 \\ R_5 - 13R_2 \end{array} \begin{bmatrix} 2 & 3 & 7 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 2.

$\mathcal{L}(v_1, v_2, v_3)$ is a 2-dimensional plane in $\mathbb{R}^5$. The space can be spanned by 2 vectors. The third vector is redundant.

$\{v_1, v_2\}$ is a linearly independent efficient basis.

$\{v_1, v_2, v_3\}$ is linearly dependent.



Two vectors determine a plane. This plane is the set spanned by them.

Suppose v_1 & v_2 are independent.

~~& v_3~~ If v_3 is on the plane determined by v_1 & v_2, then $\{v_1, v_2, v_3\}$ is linearly dependent.